Similarity and topologies generated by iterations of functions

Małgorzata Filipczak and Małgorzata Terepeta

Abstract. We examine families of sets with nonempty interiors with respect to topologies generated by functions. We also study properties of topologies generated by iterations of the functions and consider the similarity of such topologies.

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1. Introduction

On the same nonempty set X we can define different topologies and compare them by inclusion or take into considerations some other properties such that separating or countability axioms, compactness, metrizability and so on. We often identify the topological spaces when they are homeomorphic. It is known that topologies with different separating axioms can determine the same family of sets with nonempty interior. In this paper, after [2], we will say that two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ defined on the same set are *similar* if the families of sets with nonempty interior with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal. Then we will write $\mathcal{T}_1 \sim \mathcal{T}_2$. The example of similar topologies with quite different properties are the natural topology on reals $\mathcal{T}_{nat}$ and the Sorgenfrey topology generated by the base consisting of intervals of the form $[a, b)$. Another example of nonhomeomorphic similar topologies are topologies generated by lower density operators with respect to σ-algebra $\mathcal{L}$ of Lebesgue measurable sets and σ-ideal $\mathcal{N}$ of null sets.

M. Filipczak
Department of Mathematics and Computer Science, Łódź University, ul. Stefana Banacha 22, 90-238 Łódź, Poland, e-mail: malfil@math.uni.lodz.pl

M. Terepeta
Center of Mathematics and Physics, Łódź University of Technology, al. Politechniki 11, 90-924 Łódź, Poland, e-mail: terepeta@p.lodz.pl

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Recall that a *lower density operator* $\Phi : \mathcal{L} \to \mathcal{L}$ satisfies conditions:

(1) $\Phi(\varnothing) = \varnothing$, $\Phi(\mathbb{R}) = \mathbb{R}$;
(2) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for all $A, B \in \mathcal{L}$;
(3) if $A \triangle B \in \mathcal{N}$ then $\Phi(A) = \Phi(B)$;
(4) $A \triangle \Phi(A) \in \mathcal{N}$ for any $A \in \mathcal{L}$

and the family $\mathcal{T}_\Phi = \{E \in \mathcal{L} : E \subset \Phi(E)\}$ is a topology called an *abstract density topology generated by* Φ. The most known is the *classical density topology* $\mathcal{T}_d$ generated by the operator Φ_d defined as follows: for each $E \in \mathcal{L}$

$$x \in \Phi_d(E) \iff \lim_{h \to 0+} \frac{\lambda(E \cap [x-h, x+h])}{2h} = 1.$$

There are a lot of non-homeomorphic abstract density topologies (compare [1]), but properties of such topologies are quite similar (see for example [13, Sections 6.B and 6.E]). In particular, for any abstract density topology $\mathcal{T}_\Phi$ and any $A \subset \mathbb{R}$, its interior is given by the formula $int_{\mathcal{T}_\Phi}(A) = A \cap \Phi(B)$, where B is a measurable kernel of A. Hence any set of positive measure has nonempty interior in each topology generated by a lower density operator.

In [13, Section 6.D] there is presented the *superdensity topology* $\mathcal{T}_s$ generated by the operator defined for each $E \in \mathcal{L}$ in the following way

$$\Phi_s(E) = \left\{ x \in \mathbb{R} : \lim_{h \to 0+} \frac{\lambda([x-h, x+h] \setminus E)}{h^2} = 0 \right\}.$$

The operator Φ_s is not a lower density operator because there is a measurable set A of positive measure such that $\Phi_s(A) = \varnothing$ (compare [13, Example 6.27]). However, the family $\mathcal{T}_s = \{E \in \mathcal{L} : E \subset \Phi_s(E)\}$ is a topology (coarser then the density topology $\mathcal{T}_d$).

Using the notion of similarity we can say that any topology generated by lower density operator is similar to $\mathcal{T}_d$ but $\mathcal{T}_s$ and $\mathcal{T}_d$ are not similar.

In the next section we will define certain density-type topologies called f-density topologies. The topologies $\mathcal{T}_d$ and $\mathcal{T}_s$ are the examples of f-density topologies – the first is generated by the function $f(x) = x$ and the second by the function $f(x) = x^2$. In the paper we will consider topologies generated by functions and compare the families of sets with nonempty interiors in such topologies. We will pay special attention to the topologies generated by iterations of a fixed function.

2. Topologies generated by functions and similarity

Let $\mathcal{A}$ be the family of all nondecreasing functions $f\colon (0,\infty) \to (0,\infty)$ such that $\lim_{x \to 0^+} f(x) = 0$ and $\liminf_{x \to 0^+} \frac{f(x)}{x} < \infty$. Fix a function $f \in \mathcal{A}$. We say that $x \in \mathbb{R}$ a *right-hand f-density point* of a measurable set E if

$$\lim_{h \to 0+} \frac{\lambda((x, x+h) \setminus E)}{f(h)} = 0.$$

Analogously we define a left-hand f-density point of E and say that x is a f-density point of E if it is a left-hand and a right-hand f-density point of E. Denote by $\Phi_f(E)$ the set of f-density points of a measurable set E. This notion is a generalization of classical density ([14]), $\langle s\rangle$-density ([9, 6]) and ψ-density ([12]).

Note, that the condition $\liminf_{x\to 0+} \frac{f(x)}{x} < \infty$ is crucial. Indeed, if $\liminf_{x\to 0+} \frac{f(x)}{x} = \infty$ then, for any $x \in \mathbb{R}$ and any measurable set E

$$\frac{\lambda\left((x, x+h) \setminus E\right)}{f(h)} \leqslant \frac{h}{f(h)} \underset{h\to 0+}{\longrightarrow} 0.$$

For any $f \in \mathcal{A}$ the family $\mathcal{T}_f := \{E \in \mathcal{L} : E \subset \Phi_f(E)\}$ forms a topology (called f-density topology or topology generated by a function f) which is finer than the natural topology on $\mathbb{R}$ ([3]).

It is not difficult to check that, for any $f \in \mathcal{A}$, the operator Φ_f satisfies conditions (1)–(3) of a lower density operator. Fulfillment of the condition (4) depends on the value of $\liminf_{x\to 0+} \frac{f(x)}{x}$. If this limit is positive then almost all points of a measurable set are its f-density points ([6]). Consequently, the interior of a measurable set E is equal to $E \cap \Phi_f(E)$ and any set $A \subset \mathbb{R}$ has nonempty $\mathcal{T}_f$-interior if and only if A has a positive inner measure. Moreover, $(\mathbb{R}, \mathcal{T}_f)$ is a completely regular Baire space (see [6]). The classical density topology $\mathcal{T}_d$ is a topology of this kind and any topology $\mathcal{T}_f$ with $\liminf_{x\to 0+} \frac{f(x)}{x} > 0$ is similar to $\mathcal{T}_d$.

Suppose now that $f \in \mathcal{A}$ and $\liminf_{x\to 0+} \frac{f(x)}{x} = 0$. In [6] it is shown that there exist a closed set F of positive measure which has no f-density points and a closed set F_1 such that $\Phi_f(F_1) = \{0\}$ (see also [8]). Therefore Φ_f is not a lower density operator and is not idempotent. Moreover, in general, for each $E \in \mathcal{L}$, $E \cap \Phi_f(E)$ need not be included in $int_{\mathcal{T}_f}(E)$ (for example $\Phi_f(F_1) = \{0\}$ and $int_{\mathcal{T}_f}(F_1) = \varnothing$). The situation changes when $\lambda(\Phi_f(E) \triangle E) = 0$. By condition (2) from the definition of lower density operator we obtain $\Phi_f(E) \cap E \subset \Phi_f(E) = \Phi_f(\Phi_f(E) \cap E)$ and $\Phi_f(E) \cap E \in \mathcal{T}_f$. Hence

Remark 2.1. *For each $E \in \mathcal{L}$, if $\lambda(\Phi_f(E) \triangle E) = 0$, then $int_{\mathcal{T}_f}(E) = \Phi_f(E) \cap E$.*

The superdensity topology $\mathcal{T}_s$ is a topology of this kind. Obviously, no f-density topology $\mathcal{T}_f$ with $\liminf_{x\to 0+} \frac{f(x)}{x} = 0$ is similar to $\mathcal{T}_d$. There is a natural question whether such topologies are similar to $\mathcal{T}_s$. The answer is negative.

Proposition 2.2. *For any function $f \in \mathcal{A}$ with $\liminf_{x\to 0+} \frac{f(x)}{x} = 0$ there is a function $f_1 \in \mathcal{A}$ satisfying $\lim_{x\to 0+} \frac{f_1(x)}{x} = 0$ such that the topologies $\mathcal{T}_f$ and $\mathcal{T}_{f_1}$ are not similar.*

Proof. From Theorem 2 from [5] it follows that there exists a closed set $F \subset [0,1]$ of positive measure such that no point of F is an f-density point of F. Clearly $int_{\mathcal{T}_f}(F) = \varnothing$. By Theorem 3 in [11] there exists a continuous nondecreasing function $\psi : (0,\infty) \to (0,\infty)$ such that $\lim_{h\to 0+} \psi(h) = 0$ and

$$\lim_{h\to 0+} \frac{\lambda\left((x, x+h) \setminus F\right)}{h \cdot \psi(h)} = 0 \quad \text{and} \quad \lim_{h\to 0+} \frac{\lambda\left((x-h, x) \setminus F\right)}{h \cdot \psi(h)} = 0$$

for almost all $x \in F$. The function $f_1(x) = x \cdot \psi(x)$ belongs to $\mathcal{A}$, $\lim_{x\to 0+} \frac{f_1(x)}{x} = 0$ and almost all points of F are f_1-density points of F. Therefore $\Phi_{f_1}(F) = \Phi_{f_1}(\Phi_{f_1}(F) \cap F)$ and $int_{\mathcal{T}_{f_1}}(F) = \Phi_{f_1}(F) \cap F$. Hence $int_{\mathcal{T}_{f_1}}(F) \neq \varnothing$. □

In [7] there is presented a convenient description of "position" of topologies generated by functions relative to the classical density topology $\mathcal{T}_d$.

Theorem 2.3 (Corollary 1, [7]). *Let $f \in \mathcal{A}$.*

(a) $\mathcal{T}_f = \mathcal{T}_d \iff 0 < \liminf_{x\to 0+} \frac{f(x)}{x} \leqslant \limsup_{x\to 0+} \frac{f(x)}{x} < \infty$.

(b) $\mathcal{T}_f \subsetneqq \mathcal{T}_d \iff 0 = \liminf_{x\to 0+} \frac{f(x)}{x} \leqslant \limsup_{x\to 0+} \frac{f(x)}{x} < \infty$.

(c) $\mathcal{T}_d \subsetneqq \mathcal{T}_f \iff 0 < \liminf_{x\to 0+} \frac{f(x)}{x} < \limsup_{x\to 0+} \frac{f(x)}{x} = \infty$.

(d) $\mathcal{T}_f \nsubseteq \mathcal{T}_d$ *and* $\mathcal{T}_d \nsubseteq \mathcal{T}_f \iff 0 = \liminf_{x\to 0+} \frac{f(x)}{x} < \limsup_{x\to 0+} \frac{f(x)}{x} = \infty$.

We will describe in a similar manner similarity of topologies $\mathcal{T}_f$ and $\mathcal{T}_d$. Let us start from a simple property of similar topologies.

Remark 2.4. *If $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are topologies defined on the same set X, $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_3$ and $\mathcal{T}_1$ is not similar to $\mathcal{T}_2$, then $\mathcal{T}_1$ is not similar to $\mathcal{T}_3$.*

Theorem 2.5. *Let $f \in \mathcal{A}$.*

(a) $\mathcal{T}_d \subset \mathcal{T}_f \Leftrightarrow \mathcal{T}_d \sim \mathcal{T}_f$.

(b) If $\mathcal{T}_f \subsetneqq \mathcal{T}_d$ then there is a sequence $(f_n)_{n=1}^{\infty}$ of functions from the family $\mathcal{A}$ such that $(\mathcal{T}_{f_n})_{n=1}^{\infty}$ is an increasing sequence of topologies such that $\mathcal{T}_f \subsetneqq \mathcal{T}_{f_n} \subsetneqq \mathcal{T}_d$ for any $n \in \mathbb{N}$ and $\mathcal{T}_{f_i}$ is not similar to $\mathcal{T}_{f_j}$ for any $i \neq j$.

Proof. If $\mathcal{T}_d \subset \mathcal{T}_f$ then $\liminf_{x\to 0^+} \frac{f(x)}{x} > 0$ and Φ_f is a lower density operator. Consequently, $\mathcal{T}_d \sim \mathcal{T}_f$. If $\mathcal{T}_d$ is not included in $\mathcal{T}_f$ then f satisfies (b) or (d) from Theorem 2.3. In both cases $\liminf_{x\to 0^+} \frac{f(x)}{x} = 0$. Therefore, by [6, Theorem 8] there exists a closed set F of positive measure with empty $\mathcal{T}_f$-interior. Hence $\mathcal{T}_f$ is not similar to $\mathcal{T}_d$.

Suppose now that $\mathcal{T}_f \subsetneqq \mathcal{T}_d$. Repeating considerations from the proof of Proposition 2.2 we can find a closed set $F_1 \subset [0,1]$ of positive measure such that $int_{\mathcal{T}_f}(F_1) = \varnothing$ and a function $g \in \mathcal{A}$ with $\lim_{x\to 0+} \frac{g(x)}{x} = 0$ and such that $int_{\mathcal{T}_g}(F_1) \neq \varnothing$. The function $f_1(x) := \max\{f(x), g(x)\}$ belongs to $\mathcal{A}$. It is evident that if $x \in \Phi_g(F_1)$ then $x \in \Phi_{f_1}(F_1)$. It follows that $int_{\mathcal{T}_{f_1}}(F_1) \neq \varnothing$, $\mathcal{T}_f \subset \mathcal{T}_{f_1}$ and $\mathcal{T}_f$ is not similar to $\mathcal{T}_{f_1}$. Moreover, $\liminf_{x\to 0^+} \frac{f_1(x)}{x} = 0$ and $\limsup_{x\to 0^+} \frac{f_1(x)}{x} < \infty$. Hence $\mathcal{T}_f \subsetneqq \mathcal{T}_{f_1} \subsetneqq \mathcal{T}_d$.

We now proceed by induction and find a sequence $\{f_n\}_{n=1}^{\infty}$ of functions from the family $\mathcal{A}$ such that, for any $n \in N$, $\mathcal{T}_{f_n} \subsetneqq \mathcal{T}_{f_{n+1}} \subsetneqq \mathcal{T}_d$ and $\mathcal{T}_{f_n}$ is not similar to $\mathcal{T}_{f_{n+1}}$. By Remark 2.4, the sequence $(f_n)_{n=1}^{\infty}$ satisfies all conditions formulated in (b). □

Corollary 2.6. *There exists an increasing sequence $(\mathcal{T}_n)_{n=1}^{\infty}$ of topologies generated by functions, such that $\mathcal{T}_s \subsetneqq \mathcal{T}_n \subsetneqq \mathcal{T}_d$ for any $n \in \mathbb{N}$ and $\mathcal{T}_i$ is not similar to $\mathcal{T}_j$ for any $i \neq j$.*

At the end of this section let us remind one more topology connected with σ-algebra $\mathcal{L}$ and σ-ideal $\mathcal{N}$ - the Hashimoto topology

$$\mathcal{T}^* := \{G \setminus P : G \in \mathcal{T}_{nat}, P \in \mathcal{N}\}.$$

Since $\mathcal{T}^* \subsetneqq \mathcal{T}_f$ for any $f \in \mathcal{A}$ and $\bigcap_{f\in\mathcal{A}} \mathcal{T}_f = \mathcal{T}^*$([3]), we can treat the Hashimoto topology as a "lower bound" of all f-density topologies.

It is easy to see that $\mathcal{T}^*$ is not similar to $\mathcal{T}_{nat}$ (for example, the set of all irrational numbers is open in $\mathcal{T}^*$ and has empty interior in $\mathcal{T}_{nat}$). Cantor-like set of positive measure is an example of a set with nonempty interior in $\mathcal{T}_d$ (and all $\mathcal{T}_f$ with $\liminf_{x\to 0^+} \frac{f(x)}{x} > 0$) and empty interior in $\mathcal{T}^*$. It is much more difficult to check similarity between $\mathcal{T}_f$ with $\liminf_{x\to 0^+} \frac{f(x)}{x} = 0$ and $\mathcal{T}^*$. We will discuss this problem in the last section.

3. Topologies generated by iteration of a function

We will consider iterations: $f^1 = f$, $f^k = f\left(f^{k-1}\right)$ for $k \in \mathbb{N}$. It is evident that if $f \in \mathcal{A}$, then for any $k \in \mathbb{N}$ the function f^k is nondecreasing and tending to zero when x approaches zero. The property that the lower limit of the fraction $\frac{f^k(x)}{x}$ is finite may not be fulfilled. Our first problem is to examine when f^k, $k \in \mathbb{N}$, belongs to the family $\mathcal{A}$. Next we shall check if the properties of topologies generated by a function and its iteration are the same. First notice that

Proposition 3.1. *For each function $f \in \mathcal{A}$, we have the following:*

(a) If $\liminf_{x\to 0+} \frac{f(x)}{x} < 1$, then $f^k \in \mathcal{A}$ for any $k \in \mathbb{N}$.

(b) If $\limsup_{x\to 0+} \frac{f(x)}{x} < \infty$, then $f^k \in \mathcal{A}$ for any $k \in \mathbb{N}$.

Proof. Assume that $\liminf_{x\to 0+} \frac{f(x)}{x} < 1$. Then there exists a decreasing sequence $(x_n)_{n\in\mathbb{N}}$ tending to zero such that $\frac{f(x_n)}{x_n} < 1$, for $n \in \mathbb{N}$. From monotonicity of f we obtain $f^k(x_n) \leqslant \ldots \leqslant f(x_n) < x_n$, for $n \in \mathbb{N}$, and

$$\liminf_{x\to 0+} \frac{f^k(x)}{x} \leqslant \lim_{n\to\infty} \frac{f^k(x_n)}{x_n} \leqslant 1.$$

To prove the condition (b), first we will show that if $\limsup_{x\to 0+} \frac{f(x)}{x} < \infty$ then $\limsup_{x\to 0+} \frac{f^2(x)}{f(x)} < \infty$. From the assumption there exist numbers $M > 0$ and $\delta_1 > 0$ such that $\frac{f(x)}{x} < M$ for any $x \in (0, \delta_1)$. As $\lim_{x\to 0+} f(x) = 0$, so there is $\delta_2 > 0$ such that $f(x) < \delta_1$ for any $x \in (0, \delta_2)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in (0, \delta)$ we have

$$\frac{f(x)}{x} < M \text{ and } f(x) < \delta_1.$$

Hence for each $x \in (0,\delta)$, $\frac{f^2(x)}{f(x)} < M$ and $\frac{f(x)}{x} < M$, so

$$f^2(x) < Mf(x) < M^2 x.$$

Therefore,

$$\limsup_{x\to 0+} \frac{f^2(x)}{x} \leqslant M^2 < \infty.$$

Analogously, by the induction we can proof that

$$\limsup_{x\to 0+} \frac{f^k(x)}{x} < \infty$$

for each $k \in \mathbb{N}$. □

It is interesting that we can neither change the number 1 from the condition (a) nor consider the weaker inequality.

Example 3.2. *There exists a function $f \in \mathcal{A}$ such that $\liminf_{x\to 0+} \frac{f(x)}{x} = 1$ and $f^2 \notin \mathcal{A}$.*

Take a sequence $(a_n)_{n\in\mathbb{N}}$ decreasing to zero and such that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty$. Put $f(x) = a_n$ for $x \in [a_{n+1}, a_n)$. Then $\liminf_{x\to 0+} \frac{f(x)}{x} = 1$, so $f \in \mathcal{A}$. Simultaneously $f^2(x) = a_n$ for $x \in [a_{n+2}, a_{n+1})$. For any $x < a_2$ there is $n \in \mathbb{N}$ such that $x \in [a_{n+2}, a_{n+1})$. Hence $\frac{f^2(x)}{x} \geqslant \frac{a_n}{a_{n+1}}$ and $\liminf_{x\to 0+} \frac{f^2(x)}{x} \geqslant \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty$. So $f^2 \notin \mathcal{A}$. □

It is evident that different functions can generate the same topologies. However, if we have equal topologies generated by functions we can not predict the behavior of topologies generated by iterations of these functions.

Example 3.3. *There exist functions $f, g \in \mathcal{A}$ such that $\mathcal{T}_f = \mathcal{T}_g$ and $f^k \notin \mathcal{A}$ but $g^k \in \mathcal{A}$ for any $k \in \mathbb{N}$.*

Take a sequence $(a_n)_{n\in\mathbb{N}}$ and the function f as in Example 3.2. Put $g(x) = a_n$ for $x \in (a_{n+1}, a_n]$. Obviously $g^k = g$ for any $k \in \mathbb{N}$. It is not difficult to check that $\mathcal{T}_f = \mathcal{T}_g$ (it also follows immediately from [4], Theorem 2). We have shown in the previous example that $f^2 \notin \mathcal{A}$. Analogously $f^k \notin \mathcal{A}$ for any $k \in \mathbb{N}$. □

Example 3.4. *There exist functions $f, g \in \mathcal{A}$ such that $\mathcal{T}_f = \mathcal{T}_g$ and $\mathcal{T}_{g^k} \subsetneqq \mathcal{T}_{f^k}$ for any $k \in \mathbb{N}$.*

Take a sequence $(a_n)_{n\in\mathbb{N}}$ as in Example 3.2. Put $f(x) = a_{n+1}$ for $x \in [a_{n+1}, a_n)$ and $g(x) = a_{n+1}$ for $x \in (a_{n+1}, a_n]$. Then $\mathcal{T}_f = \mathcal{T}_g$ and f^k, $g^k \in \mathcal{A}$, $k \in \mathbb{N}$. Notice that and any $k \in \mathbb{N}$, $f^k(x) = f(x)$ and $g^k(x) = a_{n+k}$ for $x \in (a_{n+1}, a_n]$. Therefore $\lim_{x\to 0+} \frac{f^k(x)}{g^k(x)} = \infty$ and from [4], Corollary 1, we obtain $\mathcal{T}_{g^k} \subsetneqq \mathcal{T}_{f^k}$, $k \in \mathbb{N}$. □

As it is described in Theorem 2.3, the family of all f-density topologies can be splitted into 4 subfamilies. One can ask if it is possible, that there exists a number $k \in \mathbb{N}$ such that the topology $\mathcal{T}_{f^k}$ belongs to different subfamily than $\mathcal{T}_f$? To answer this question we will need the following technical proposition.

Proposition 3.5. *Let $f, f^k \in \mathcal{A}$, $k \in \mathbb{N}$.*

(a) $\liminf\limits_{x\to 0+} \frac{f(x)}{x} = 0 \iff \liminf\limits_{x\to 0+} \frac{f^k(x)}{x} = 0.$

(b) $\limsup\limits_{x\to 0+} \frac{f(x)}{x} = \infty \iff \limsup\limits_{x\to 0+} \frac{f^k(x)}{x} = \infty.$

Proof. Assume that $\liminf\limits_{x\to 0+} \frac{f(x)}{x} = 0$. Then there exists a decreasing sequence $(x_n)_{n\in\mathbb{N}}$ tending to 0 such that

$$\liminf_{x\to 0+} \frac{f(x)}{x} = \lim_{n\to\infty} \frac{f(x_n)}{x_n} = 0.$$

So there is a number $n_0 \in \mathbb{N}$ such that $\frac{f(x_n)}{x_n} < 1$ for $n > n_0$. Hence $f^k(x_n) \leqslant f^{k-1}(x_n) \leqslant \ldots \leqslant f(x_n) < x_n$ and

$$\liminf_{x\to 0+} \frac{f^k(x)}{x} \leqslant \lim_{n\to\infty} \frac{f^k(x_n)}{x_n} \leqslant \lim_{n\to\infty} \frac{f(x_n)}{x_n} = 0.$$

Now we assume that $\liminf\limits_{x\to 0+} \frac{f(x)}{x} > 0$. In the same way as in Property 3.1 we can prove that $\liminf\limits_{x\to 0+} \frac{f^k(x)}{x} > 0$, for any $k \in \mathbb{N}$.

The proof of the condition (b) is analogous. □

The next theorem is a simple consequence of the above proposition, Theorem 2.3 and the fact that if $f(x) \leqslant g(x)$ for $x > 0$, then $\mathcal{T}_f \subset \mathcal{T}_g$.

Theorem 3.6. *Let $f \in \mathcal{A}$.*

(a) If $0 < \liminf\limits_{x\to 0+} \frac{f(x)}{x} \leqslant \limsup\limits_{x\to 0+} \frac{f(x)}{x} < \infty$, *then* $\mathcal{T}_f = \mathcal{T}_{f^k} = \mathcal{T}_d$.

(b) If $0 = \liminf\limits_{x\to 0+} \frac{f(x)}{x} \leqslant \limsup\limits_{x\to 0+} \frac{f(x)}{x} < \infty$, *then* $\mathcal{T}_{f^k} \subset \mathcal{T}_f \subsetneq \mathcal{T}_d$.

(c) If $0 < \liminf\limits_{x\to 0+} \frac{f(x)}{x} < \limsup\limits_{x\to 0+} \frac{f(x)}{x} = \infty$, *then* $\mathcal{T}_d \subsetneq \mathcal{T}_f$ *and* $\mathcal{T}_d \subsetneq \mathcal{T}_{f^k}$

(d) If $0 = \liminf\limits_{x\to 0+} \frac{f(x)}{x} < \limsup\limits_{x\to 0+} \frac{f(x)}{x} = \infty$, *then* $\mathcal{T}_{f^k} \subset \mathcal{T}_f$ *but* $\mathcal{T}_f \not\subseteq \mathcal{T}_d$ *and* $\mathcal{T}_{f^k} \not\subseteq \mathcal{T}_d$.

In general we do not know what is the relation between $\mathcal{T}_f$ and $\mathcal{T}_{f^k}$ in cases (c) and (d).

Topologies generated by functions can not be invariant under multiplication by nonzero numbers. It depends on the condition which we call $\triangle_2$, because it is very similar to the condition $\triangle_2$ considered in the theory of Orlicz spaces.

Definition 3.7 ([5])**.** *We will say that $f \in \mathcal{A}$ fulfills $\triangle_2$ condition ($f \in \triangle_2$) if*

$$\limsup_{x\to 0^+} \frac{f(2x)}{f(x)} < \infty. \qquad (\triangle_2)$$

The condition $\triangle_2$ is equivalent to the condition $\limsup\limits_{x\to 0^+} \frac{f(\alpha x)}{f(x)} < \infty$ for any number $\alpha > 0$ ([5], Proposition 3). Observe, that for any $\alpha > 0$ the function $f(x) = x^\alpha$ and any its iteration fulfill $\triangle_2$ ($f \in \mathcal{A}$ only for $\alpha \geqslant 1$).

If a function f fulfills $\triangle_2$ then the topology $\mathcal{T}_f$ is invariant under multiplication by nonzero numbers ([5], Theorem 4). If $\mathcal{T}_f \subset \mathcal{T}_d$ then $\triangle_2$ is the necessary and sufficient condition for this invariantness. If $\mathcal{T}_f \not\subset \mathcal{T}_d$ then there exists a function $g \notin \triangle_2$ such that $\mathcal{T}_f = \mathcal{T}_g$ ([5], Theorem 6). One can ask, if it is possible that for a function $f \in \triangle_2$ there exists a number $k \in \mathbb{N}$ such that f^k has not such property. The answer is negative due to the following theorem.

Theorem 3.8. *If $f, g \in \mathcal{A}$ fulfill the condition $\triangle_2$, then their composition $g \circ f$ also fulfills $\triangle_2$.*

Proof. Assume that $f \in \triangle_2$. Hence there exist numbers $M > 0$, $\delta_1 > 0$ such that $\frac{f(2x)}{f(x)} < M$ for $x \in (0, \delta_1)$. By the monotonicity of the function g we have

$$g(f(2x)) < g(Mf(x))$$

for each $x \in (0, \delta_1)$. Since $g \in \triangle_2$, we have that for each $\alpha > 0$, $\limsup_{x\to 0^+} \frac{g(\alpha x)}{g(x)} < \infty$. Put $\alpha = M$. Then there exist $K > 0$ and $\delta_2 > 0$ such that $\frac{g(Mx)}{g(x)} < K$ for $x \in (0, \delta_2)$. From the assumption $\lim_{x\to 0^+} f(x) = 0$ it follows that there exists $\delta_3 > 0$ such that $f(x) < \delta_2$ whenever $x \in (0, \delta_3)$. Hence for $\delta = \min(\delta_1, \delta_2, \delta_3)$ and $x \in (0, \delta)$,

$$\frac{g(f(2x))}{g(f(x))} \leqslant \frac{g(Mf(x))}{g(f(x))} < K.$$

Therefore,

$$\limsup_{x\to 0+} \frac{g(f(2x))}{g(f(x))} < \infty.$$

□

Corollary 3.9. *If $f \in \triangle_2$, then $f^k \in \triangle_2$ for any $k \in \mathbb{N}$.*

It is easy to check that if $f \in \mathcal{A}$ and

$$0 < \liminf_{x\to 0+} \frac{f(x)}{x} \leqslant \limsup_{x\to 0+} \frac{f(x)}{x} < \infty, \tag{1}$$

then $f \in \triangle_2$ if and only if $f^k \in \triangle_2$. Moreover, if (1) holds then $\mathcal{T}_f = \mathcal{T}_{f^k} = \mathcal{T}_d$ so this case is not interesting, because the density topology is invariant under multiplication by nonzero numbers. The next examples show that if one of the inequalities of (1) is not fulfilled we do not have the equivalence $f \in \triangle_2 \iff f^k \in \triangle_2$. We will use the functions similar to the functions defined in [7], Lemma 1.

Example 3.10. *There exists a function $g \in \mathcal{A}$ such that $0 = \liminf_{x\to 0+} \frac{g(x)}{x} < \limsup_{x\to 0+} \frac{g(x)}{x} < \infty$, $g \notin \triangle_2$ and $g^2 \in \triangle_2$.*

Let $a_n = \sqrt{b_n b_{n+1}}$ for $n \in \mathbb{N}$, where $(b_n)_{n\geqslant 0}$ is a strictly decreasing sequence tending to zero such that $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 0$. Put

$$g(x)=\begin{cases}\frac{x^2}{b_n} & \text{for } x\in(\sqrt{a_nb_n},b_n),\\ b_{n+1} & \text{for } x\in[b_{n+1},\sqrt{a_nb_n}],\\ b_1 & \text{for } x\geqslant b_1.\end{cases}$$

Obviously, g is nondecreasing, $g(x)\leqslant 1$ for any $x>0$ and $\lim\limits_{x\to 0+} g(x)=0$. Moreover,

$$\liminf_{x\to 0+}\frac{g(x)}{x}\leqslant\lim_{n\to\infty}\frac{g(a_n)}{a_n}=\lim_{n\to\infty}\sqrt{\frac{b_{n+1}}{b_n}}=0.$$

Hence $g\in\mathcal{A}$ and $\limsup\limits_{x\to 0+}\frac{g(x)}{x}<\infty$. It does not fulfil condition $\triangle_2$. Indeed, let $x>0$. Then $x\in[b_{n+1},b_n)$ for a certain $n\in\mathbb{N}$. Therefore,

$$\limsup_{x\to 0+}\frac{g(2x)}{g(x)}\geqslant\lim_{n\to\infty}\frac{g(2\sqrt{a_nb_n})}{g(\sqrt{a_nb_n})}\geqslant\lim_{n\to\infty}\sqrt{\frac{b_n}{b_{n+1}}}=\infty.$$

From the definition of g we obtain

$$g^2(x)=\begin{cases}\frac{x^4}{b_n^3} \text{ for } x\in[\sqrt{a_nb_n},b_n),\\ b_{n+1} \text{ for } x\in[b_{n+1},\sqrt{a_nb_n}),\\ b_1 \text{ for } x\geqslant b_1.\end{cases}$$

From Proposition 3.1 it follows that $g^2\in\mathcal{A}$. We will show that $g^2\in\triangle_2$. Notice, that for $x\geqslant\sqrt{a_nb_n}$ we have $g^2(x)\leqslant\frac{x^4}{b_n^3}$. Take $x>0$. Then there is natural number n such that $x\in(b_{n+1},b_n]$. If $x,2x\in[\sqrt{a_nb_n},b_n)$ then $g^2(x)=\frac{x^4}{b_n^3}$ and $g(2x)\leqslant\frac{(2x)^4}{b_n^3}$. Hence $\frac{g^2(2x)}{g^2(x)}\leqslant 16$. If $x,2x\in[b_{n+1},\sqrt{a_nb_n})$ then $\frac{g^2(2x)}{g^2(x)}=1$. If $x\in[\sqrt{a_nb_n},b_n)$ and $2x\geqslant b_n$ then $\frac{g^2(2x)}{g^2(x)}=\frac{g^2(2x)}{g^2(b_n)}\cdot\frac{g^2(b_n)}{g^2(x)}\leqslant\frac{g^2(2b_n)}{g^2(b_n)}\cdot 1\leqslant 16$. Hence $g^2\in\triangle_2$. □

Example 3.11. *There exists a function $g\in\mathcal{A}$ such that $0<\liminf\limits_{x\to 0+}\frac{g(x)}{x}<\limsup\limits_{x\to 0+}\frac{g(x)}{x}=\infty$, $g\notin\triangle_2$ and $g^2\in\triangle_2$.*

Let $(a_n)_{n\geqslant 0}$ be a strictly decreasing sequence tending to zero such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=0$ and $b_n=\sqrt{a_na_{n+1}}$, $n\in\mathbb{N}$. Put

$$g(x)=\begin{cases}\frac{x^2}{a_n} & \text{for } x\in(a_n,\sqrt{a_nb_n}],\\ a_{n-1} & \text{for } x\in(\sqrt{a_nb_n},a_{n-1}],\\ a_0 & \text{for } x\geqslant a_0.\end{cases}$$

The function g belongs to $\mathcal{A}$ and it does not fulfill condition $\triangle_2$. Indeed,

$$\frac{g(2\sqrt{a_nb_n})}{g(\sqrt{a_nb_n})}\geqslant\frac{a_{n-1}}{b_n}=\sqrt{\frac{a_{n-1}}{a_n}}$$

and

$$\limsup_{x\to 0+}\frac{g(2x)}{g(x)} \geqslant \lim_{n\to\infty}\frac{g(2\sqrt{a_nb_n})}{g(\sqrt{a_nb_n})} = \infty.$$

Moreover,

$$\limsup_{x\to 0+}\frac{g(x)}{x} \geqslant \lim_{n\to\infty}\frac{g(\sqrt{a_nb_n})}{\sqrt{a_nb_n}} = \lim_{n\to\infty}\sqrt[4]{\frac{a_{n+1}}{a_n}} = \infty.$$

From the definition of g we obtain

$$g^2(x) = \begin{cases} \frac{x^4}{a_n^3} \text{ for } x \in (a_n, \sqrt{a_nb_n}], \\ a_n \text{ for } x \in (\sqrt{a_{n+1}b_{n+1}}, a_n], \\ a_1 \text{ for } x \geqslant a_1. \end{cases}$$

The function g^2 is continuous, $g^2 \in \mathcal{A}$. Analogously as in previous example we show that $g^2 \in \triangle_2$. □

Example 3.12. *There exists a function $g \in \mathcal{A}$ such that $0 = \liminf_{x\to 0+}\frac{g(x)}{x} < \limsup_{x\to 0+}\frac{g(x)}{x} = \infty$, $g \notin \triangle_2$ and $g^2 \in \triangle_2$.*

Let $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ be sequences from the previous example. Put

$$g(x) = \begin{cases} \frac{x^2}{a_{2n-1}} & \text{for } x \in \left[b_{2n}, \sqrt{a_{2n-1}b_{2n-2}}\right), \\ a_{2n} & \text{for } x \in \left[\sqrt{a_{2n+1}b_{2n+1}}, b_{2n}\right), \\ a_0 & \text{for } x \geqslant a_1. \end{cases}$$

The function g has all required properties: belongs to $\mathcal{A}$, it does not fulfill condition $\triangle_2$ and $0 = \liminf_{x\to 0+}\frac{g(x)}{x} < \limsup_{x\to 0+}\frac{g(x)}{x} = \infty$. From the definition of g we obtain

$$g^2(x) = \begin{cases} \frac{x^4}{a_{2n-1}^3} & \text{for } x \in [\sqrt{a_{2n-1}b_{2n}}, \sqrt{a_{2n-1}b_{2n-1}}), \\ a_{2n} & \text{for } x \in [\sqrt{a_{2n+1}b_{2n+1}}, \sqrt{a_{2n-1}b_{2n}}), \\ a_0 & \text{for } x \geqslant \sqrt{a_1b_1}. \end{cases}$$

The function g^2 is continuous, $g^2 \in \mathcal{A}$. Analogously as in previous example we show that $g^2 \in \triangle_2$. □

In Examples 3.10 and 3.11 we have constructed the functions which fulfill the condition $\limsup_{x\to 0+}\frac{g(2x)}{g(x)} = \infty$ ($g \notin \triangle_2$), but their iterations satisfy $\triangle_2$. We can not exchange the the upper limit in this condition into the limit as it is shown by the next theorem.

Theorem 3.13. *Let $f, f^k \in \mathcal{A}$. If $\lim_{x\to 0+}\frac{f(2x)}{f(x)} = \infty$, then $f^k \notin \triangle_2$ for any $k \in \mathbb{N}$.*

Proof. From the assumption, $\lim_{n\to\infty}\frac{f(2x_n)}{f(x_n)} = \infty$ for any sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = 0$, in particular

$$\lim_{n\to\infty}\frac{f(2f(x_n))}{f(f(x_n))} = \infty. \tag{2}$$

Fix the sequence $(x_n)_{n\in\mathbb{N}}$. There is $n_0 \in \mathbb{N}$ such that $\frac{f(2x_n)}{f(x_n)} > 2$ for $n > n_0$. Hence

$$f^2(2x_n) \geqslant f(2f(x_n)). \tag{3}$$

Therefore, for each sequence $(x_n)_{n\in\mathbb{N}}$ we obtain

$$\lim_{n\to\infty} \frac{f^2(2x_n)}{f^2(x_n)} \overset{(3)}{\geqslant} \lim_{n\to\infty} \frac{f(2f(x_n))}{f^2(x_n)} = \infty$$

and finally we obtain $\lim\limits_{x\to 0+} \frac{f^2(2x)}{f^2(x)} = \infty$, so $f^2 \notin \triangle_2$. We can prove by induction that $\lim\limits_{x\to 0+} \frac{f^k(2x)}{f^k(x)} = \infty$ for any $k \in \mathbb{N}$, which finishes the proof. □

4. Similarity between topologies generated by function and its iteration

Now we will focus on a problem: do there exist topologies generated by a function and its iteration which are not similar? Theoretically we know that there are f-density topologies which are not similar, but in practice it is not easy to indicate specific functions for which we can obtain such result. To examine the interior of sets in density topology we often construct Cantor-like sets of positive measure. It is not difficult to construct a set of this kind which has no f-density points.

Example 4.1. *There exists a set of positive measure without superdensity points.*

By induction we will define a central Cantor set $E \subset [0,1]$ of positive measure which has no f-density points for $f(x) = x^2$.

From the interval $[0,1]$ we remove concentric open interval denoted by $\big(a_1^{(1)}, b_1^{(1)}\big)$ of the length $\frac{1}{4}$. Put $E_1 = \big(a_1^{(1)}, b_1^{(1)}\big)$. Suppose that for certain $n \geqslant 2$ we have constructed the sets $E_1, E_2, \ldots, E_{n-1}$ such that $E_{n-1} = \bigcup\limits_{i=1}^{2^{n-2}} \big(a_{n-1}^{(i)}, b_{n-1}^{(i)}\big)$. The set $[0,1] \setminus \bigcup\limits_{k=1}^{n-1} E_k$ consists of 2^{n-1} closed intervals and from each such interval we remove concentric interval of the length $\frac{1}{4^n}$ which we denote by $\big(a_n^{(i)}, b_n^{(i)}\big)$, $i = 1, \ldots, 2^{n-1}$. Put $E_n = \bigcup\limits_{i=1}^{2^{n-1}} \big(a_n^{(i)}, b_n^{(i)}\big)$. Then $\lambda(E_n) = 2^{n-1} \cdot \frac{1}{4^n} = \frac{1}{2^{n+1}}$. The set $E = [0,1] \setminus \bigcup\limits_{n=1}^{\infty} E_n$ is of positive measure since $\lambda(E) = 1 - \sum\limits_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}$.

Fix $x_0 \in E$. Then for any natural n there is a number $i_n \in \{1, \ldots, 2^{n-1}\}$ such that the distance between x_0 and the interval $(a_n^{(i_n)}, b_n^{(i_n)})$ is the smallest. By putting $c_n = \frac{a_n^{(i_n)} + b_n^{(i_n)}}{2}$ we obtain a sequence $(c_n)_{n\in\mathbb{N}}$converging to x_0. Without loosing the generality, we may assume that $(c_n)_{n\in\mathbb{N}}$ is decreasing to x_0. Observe, that $c_n - x_0 \leqslant \frac{1}{2^n}$ and $\lambda\big(E' \cap [x_0, c_n]\big) \geqslant \frac{1}{2} \cdot \lambda\big((a_n^{(i_n)}, b_n^{(i_n)})\big) = \frac{1}{2} \cdot \frac{1}{4^n}$. Hence

$$\frac{\lambda\left(E'\cap[x_0,c_n]\right)}{(c_n-x_0)^2}\geqslant\frac{\frac{1}{2}\cdot\frac{1}{4^n}}{\left(\frac{1}{2^n}\right)^2}=\frac{1}{2}.$$

and $\liminf\limits_{n\to\infty}\frac{\lambda\left(E'\cap[x_0,c_n]\right)}{(c_n-x_0)^2}\geqslant\frac{1}{2}$. Therefore, x_0 is not a superdensity point of E. □

In general it is rather difficult to describe an interior of a set in f-density topologies (see [10]).

Theorem 4.2 ([10], Theorem 3 and 10)**.**

(a) For any $f\in\mathcal{A}$ and any set $A\subset\mathbb{R}$ there exists a countable ordinal $\alpha\geqslant 1$ such that $int_{\mathcal{T}_f}(A)=A\cap\Phi_f^{\alpha}(B)$ where B is a measurable kernel of A.

(b) Let $f\in\mathcal{A}$ and $\liminf\limits_{x\to 0^+}\frac{f(x)}{x}=0$. For each $n\in\mathbb{N}$ there exists a perfect nowhere dense set A such that $int_{\mathcal{T}_f}(A)\subsetneq A\cap\Phi_f^{k}(A)$ for $k<n$ and $int_{\mathcal{T}_f}(A)=A\cap\Phi_f^{n}(A)$.

However, if $\lambda\left(\Phi_f(A)\triangle A\right)=0$, then we know that the interior of A is not empty (see Remark 2.1). To answer the question from the beginning of this section, firstly we will construct Cantor-like set of positive measure which has nonempty interior in $\mathcal{T}_f$ for a power function $f(x)=x^{\alpha}$, $(\alpha>1)$ and secondly, we will show that this set has the empty interior for another power function.

Theorem 4.3. *For $f(x)=x^{\alpha}$, $\alpha>1$, there exists a perfect nowhere dense set E of positive measure such that almost every point of E is its f-density point.*

Proof. Analogously as in Example 4.1 we will define a central Cantor set $E\subset[0,1]$ of positive measure which has desired properties. Following denotations of this example, from the interval $[0,1]$ we remove concentric open interval denoted by $E_1=\left(a_1^{(1)},b_1^{(1)}\right)$ of the length $\frac{1}{2^{\alpha}}$. Suppose that for certain $n\geqslant 2$ we have the sets $E_1,E_2,\ldots,E_{n-1}$ such that $E_{n-1}=\bigcup\limits_{i=1}^{2^{n-2}}\left(a_{n-1}^{(i)},b_{n-1}^{(i)}\right)$. From each of closed intervals of the set $[0,1]\setminus\bigcup\limits_{k=1}^{n-1}E_k$ we remove concentric interval of the length $\frac{1}{2^{n-1}}\cdot\frac{1}{n2^{(2n-1)\alpha}}$ and denote it by $\left(a_n^{(i)},b_n^{(i)}\right)$, $i=1,\ldots,2^{n-1}$. By putting $E_n=\bigcup\limits_{i=1}^{2^{n-1}}\left(a_n^{(i)},b_n^{(i)}\right)$ we have $\lambda(E_n)=\frac{1}{n2^{(2n-1)\alpha}}$.

Let $E=[0,1]\setminus\bigcup\limits_{n=1}^{\infty}E_n$. Let us notice that for any $n\in\mathbb{N}$

$$\lambda(E_{n+1})<\frac{1}{2^{2\alpha}}\lambda(E_n)$$

and

$$\lambda\Big(\bigcup_{k=n+1}^{\infty}E_k\Big)\leqslant\lambda(E_n). \tag{4}$$

Take $k\in\mathbb{N}$. For any $n\geqslant k$ and $i=1,\ldots,2^{n-1}$ there exists a number $z_{n,k}^{(i)}>b_n^{(i)}$ such that

$$\frac{\lambda(E_n)}{\left(z_{n,k}^{(i)}-a_n^{(i)}\right)^{\alpha}}=\frac{1}{k}. \tag{5}$$

We can observe that if $z > z^{(i)}_{n,k}$ then

$$\frac{\lambda(E_n)}{\big(z-a^{(i)}_n\big)^{\alpha}} < \frac{1}{k}. \tag{6}$$

It is evident, that

$$z^{(i)}_{n,k} - a^{(i)}_n = \big(k\cdot\lambda(E_n)\big)^{\frac{1}{\alpha}} \tag{7}$$

and

$$z^{(i)}_{n,k} - b^{(i)}_n < \big(k\cdot\lambda(E_n)\big)^{\frac{1}{\alpha}}. \tag{8}$$

For $k\in\mathbb{N}$ define the sets

$$A_k = \bigcup_{n=k}^{\infty}\bigcup_{i=1}^{2^{n-1}} \big[b^{(i)}_n, z^{(i)}_{n,k}\big].$$

We will show that $\lambda\big(\limsup\limits_{k\to\infty} A_k\big)=0$. For any $k\in\mathbb{N}$ we have

$$\begin{aligned}\lambda(A_k) &= \sum_{n=k}^{\infty}\sum_{i=1}^{2^{n-1}}\big(z^{(i)}_{n,k}-b^{(i)}_n\big) \overset{(8)}{\leqslant} \sum_{n=k}^{\infty}\sum_{i=1}^{2^{n-1}}\big(k\cdot\lambda(E_n)\big)^{\frac{1}{\alpha}} = \\ &= \sum_{n=k}^{\infty} 2^{n-1}\cdot k^{\frac{1}{\alpha}}\cdot\frac{1}{n^{\frac{1}{\alpha}}\cdot 2^{2n-1}} \leqslant \sum_{n=k}^{\infty}\frac{1}{2^n} = \frac{1}{2^{k-1}}.\end{aligned}$$

Hence for any $m\geqslant 2$

$$\lambda\Big(\bigcup_{k=m}^{\infty} A_k\Big) \leqslant \sum_{k=m}^{\infty}\frac{1}{2^{k-1}} = \frac{1}{2^{m-2}}.$$

From the fact that the sequence $\bigcup\limits_{k=m}^{\infty} A_k$ is decreasing we obtain

$$\lambda\Big(\limsup_{k\to\infty} A_k\Big) = \lambda\Big(\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty} A_k\Big) = \lim_{m\to\infty}\lambda\Big(\bigcup_{k=m}^{\infty} A_k\Big) = 0.$$

Let $N = \limsup\limits_{k\to\infty} A_k \cup \Big(\bigcup\limits_{n=1}^{\infty}\bigcup\limits_{i=1}^{2^{n-1}}\{b^{(i)}_n\}\Big)$ and $x\in E\setminus N$. We will show that x is a left-hand f-density point of E for $f(x)=x^{\alpha}$, $\alpha>1$. Fix $\varepsilon>0$ and take $k_0\in\mathbb{N}$ such that $\frac{2}{k_0}<\varepsilon$ and $x\in E\setminus A_{k_0}$. Then there are numbers $n_0\geqslant k_0$ and $i_0\in\{1,\ldots,2^{n_0-1}\}$ such that $b^{(i_0)}_{n_0}<x$. We denote

$$t_0 = \min\Big\{\big|x-b^{(i)}_n\big| : n\leqslant n_0, i=1,\ldots,2^{n-1}, b^{(i)}_n\in\big[b^{(i_0)}_{n_0},x\big)\Big\}.$$

For any $t\in(0,t_0)$ we denote by n_t the smallest number n and by i_t the biggest number i for which $\big(a^{(i_t)}_{n_t}, b^{(i_t)}_{n_t}\big)\subset(x-t,x)$ and $b^{(i_t)}_{n_t}-a^{(i_t)}_{n_t}\leqslant b^{(i_0)}_{n_0}-a^{(i_0)}_{n_0}$. Then $n_t\geqslant n_0$ and $x>z^{(i_t)}_{n_t,k_0}$, so

$$\frac{\lambda(E_{n_t})}{\left(x-a_{n_t}^{(i_t)}\right)^{\alpha}}<\frac{1}{k_0}.$$

Since $x-t<a_{n_t}^{(i_t)}$, we have $t>x-a_{n_t}^{(i_t)}$. Hence

$$\lambda\big([x-t,x]\setminus E\big)\leqslant\lambda(E_{n_t})+\lambda\Big(\bigcup_{n=n_t+1}^{\infty}\lambda(E_n)\Big)\overset{(4)}{\leqslant}2\lambda(E_{n_t})$$

and

$$\frac{\lambda\left([x-t,x]\setminus E\right)}{t^{\alpha}}<\frac{2\lambda(E_{n_t})}{\left(x-a_{n_t}^{(i_t)}\right)^{\alpha}}<\frac{2}{k_0}<\varepsilon$$

and x is a left-hand f-density point of E.

Analogously, for any $k\in\mathbb{N}$ and $n\geqslant k$ we can define a number $y_{n,k}^{(i)}<a_n^{(i)}$ for which

$$\frac{\lambda(E_n)}{\left(b_n^{(i)}-y_{n,k}^{(i)}\right)^{\alpha}}=\frac{1}{k}$$

and the sets

$$B_k=\bigcup_{n=k}^{\infty}\bigcup_{i=1}^{2^{n-1}}\left[y_{n,k}^{(i)},a_n^{(i)}\right]$$

such that $\lambda\Big(\limsup_{k\to\infty}B_k\Big)=0$. Then any number

$$x\in E\setminus\left(\limsup_{k\to\infty}B_k\cup\Big(\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{2^{n-1}}\{a_n^{(i)}\}\Big)\right)$$

is a right-hand f-density point of E. Summarising, almost all points of E are its f-density points. □

Corollary 4.4. *For any $\alpha>1$ and function $f(x)=x^{\alpha}$ there exists a set E_α of positive measure such that $int_{\mathcal{T}_f}(E_\alpha)=E_\alpha\cap\Phi_f(E_\alpha)\neq\varnothing$.*

Proposition 4.5. *Let $f(x)=x^{\alpha}$, $g(x)=x^{3\alpha}$, $\alpha>1$. Then $\mathcal{T}_f\not\subset\mathcal{T}_g$.*

Proof. Let $f(x)=x^{\alpha}$, $\alpha>1$ and E be the set constructed in the proof of Theorem 4.3. We will show that no point of this set is its g-density point, so E has the empty interior in the topology $\mathcal{T}_g$ for $g(x)=x^{3\alpha}$.

Fix $x_0\in E$. Analogously as in Example 4.1 we have the sequence $(c_n)_{n\in\mathbb{N}}$ decreasing to x_0 such that $c_n-x_0\leqslant\frac{1}{2^n}$ and $\lambda\left(E'\cap[x_0,c_n]\right)\geqslant\frac{1}{2}\cdot\lambda\left((a_n^{(i_n)},b_n^{(i_n)})\right)=\frac{1}{2}\cdot\frac{1}{2^{n-1}}\cdot\frac{1}{n2^{(2n-1)\alpha}}$. Hence

$$\frac{\lambda\left(E'\cap[x_0,c_n]\right)}{g(c_n-x_0)}\geqslant\frac{\frac{1}{n2^{(2n-1)\alpha+n}}}{\left(\frac{1}{2^n}\right)^{3\alpha}}=\frac{2^{n(\alpha-1)+\alpha}}{n}$$

and $\lim_{n\to\infty} \frac{\lambda\left(E'\cap[x_0,c_n]\right)}{g(c_n-x_0)} = \infty$. Therefore, $x_0 \notin \Phi_g(E)$ and consequently, $E\cap\Phi_g(E) = \varnothing$. Thus E has the empty interior in $\mathcal{T}_g$. □

Observe that if the interior of a set is empty for a function $f_1(x) = x^{\alpha_1}$, then it is also empty for any function $f_2(x) = x^{\alpha_2}$ with $\alpha_2 > \alpha_1$.

Corollary 4.6. *For any function $f(x) = x^\alpha$, $\alpha > 1$, there is $n \in \mathbb{N}$ such that $\mathcal{T}_{f^n} \not\sim \mathcal{T}_f$. If $\alpha \geqslant 3$, then $\mathcal{T}_{f^2} \not\sim \mathcal{T}_f$.*

Notice that for $\alpha = 2$, Theorem 4.3 gives us the set which almost all points are its superdensity points, but Proposition 4.5 does not resolve the problem if the second iteration of $f(x) = x^2$ generates the topology which is similar or not to the superdensity topology.

Nevertheless, using Corollary 4.6 we can construct a decreasing sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ of topologies generated by functions, such that $\mathcal{T}_s \supsetneqq \mathcal{T}_n$ for any $n \in \mathbb{N}$ and $\mathcal{T}_i$ is not similar to $\mathcal{T}_j$ for any $i \neq j$. Comparing this result with Corollary 2.6 we can say that there are a lot of f-density topologies dissimilar to superdensity topology, of both kinds: the smaller and larger than $\mathcal{T}_s$.

At the end let us compare topologies generated by functions of the form $f(x) = x^\alpha$ with Hashimoto topology $\mathcal{T}^*$.

Remark 4.7. *Any nowhere dense set has an empty interior in Hashimoto topology $\mathcal{T}^*$. Consequently, by Theorem 4.3, if $f(x) = x^\alpha$ and $\alpha > 1$ then $\mathcal{T}_f$ is not similar to $\mathcal{T}^*$.*

The latter remark does not establish that no f-density topology with $\liminf_{x\to0^+} \frac{f_1(x)}{x} = 0$ is similar to $\mathcal{T}^*$. For example, the function

$$g(x) = \begin{cases} x^{\frac{1}{x}} & \text{for } x \in (0,1) \\ 1 & \text{for } x \geqslant 1 \end{cases}$$

belongs to $\mathcal{A}$ and for any $\alpha > 1$ the set E_α, constructed in Theorem 21 satisfies equality: $\Phi_g(E_\alpha) = \varnothing$. Therefore $\mathcal{T}_g$ is smaller than (and not similar to) any $\mathcal{T}_{f_\alpha}$ where $f_\alpha(x) = x^\alpha$ and $\alpha > 1$. We do not know if $\mathcal{T}_g$ is similar to $\mathcal{T}^*$.

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